Will do steps ② & ③ of:

$$\begin{array}{c} D^b(kh) \\ \wr\ ① \\ D^b(kE) \\ \wr\ ② \\ D^f(K) \\ \wr\ ③ \\ D^f(\Lambda) \\ \wr\ 4 \\ D^f(S) \end{array}$$

Setup: $E := (\mathbb{Z}/p)^{\times r}$, elem. abelian. k field, char $k = p$, $p \geq 3$

($p = 2$: later...)

$\wr$

$H^*(E,k) = k[\xi_1, \dots, \xi_r, \theta_1, \dots, \theta_r]$, $|\xi_i| = 1$, $|\theta_i| = 2$.

$M \in kE\text{-mod}$, $V(M) = V\left(\operatorname{ann}_{k[\theta]} \operatorname{Ext}^*_{kE}(k, M)\right) \subseteq \mathbb{A}^r_k$

Thm: $\left\{\begin{array}{c}\text{thick subcats.}\\ \text{of } D^f(kE)\end{array}\right\} \underset{\underset{T}{\longleftarrow}}{\overset{1-1}{\underset{V}{\overset{\cong}{\longleftrightarrow}}}} \left\{\begin{array}{c}\text{conical spec. closed}\\ \text{subsets of } \mathbb{A}^r_k\end{array}\right\}$ ← closed under scalar multipl.

GOAL:

$(*)\ \|$ $\qquad\qquad\qquad\qquad$ $\|$ ← Hopkins–Neeman

$\left\{\begin{array}{c}\text{thick subcats}\\ \text{of } D^f(\Lambda)\end{array}\right\} \underset{\text{BGG}}{\cong} \left\{\begin{array}{c}\text{thick subcats}\\ \text{of } D^f(S)\end{array}\right\}$

GOAL: 1) prove $(*)$

2) show V, T has promised form!

<u>Main tool</u>: Koszul complexes ($\in$ dga's)

- Q comm. ring, $f \in Q$

$\mathrm{Kos}_Q(f) := (0 \to \underset{1}{Q \cdot e} \xrightarrow{f} \underset{0}{Q} \to 0 \dots)$ a dga with $e^2 = 0$ (homological grading)

For $\underbrace{f_1, \dots, f_r}_{=: \underline{f}} \in Q$, set:

$$\mathrm{Kos}_Q(\underline{f}) := \mathrm{Kos}_Q(f_1) \otimes \dots \otimes \mathrm{Kos}_Q(f_r)$$

again as a dga.

Ex: $\mathrm{Kos}_Q(f_1, f_2) = \left(0 \longrightarrow Q \cdot e_1 \wedge e_2 \xrightarrow{\begin{bmatrix} -f_2 \\ f_1 \end{bmatrix}} \begin{matrix} Qe_1 \\ \oplus \\ Qe_2 \end{matrix} \xrightarrow{[f_1 \ f_2]} Q \longrightarrow 0 \right)$

- Another way to see this:

  $K := \mathrm{Kos}_Q(\underline{f})$, $K^{\natural} = \Lambda^{\bullet}(V)$, V free Q-module with basis $e_1, \dots, e_r$

  Add the differential: $d(e_i) = f_i$

- So have → an inductive def.
  → a direct def. as underlying gr. mod + diff'l.

- $kE \cong \dfrac{k[z_1, \dots, z_r]}{(z_1^p, \dots, z_r^p)} =: R$

  $Q := k[z_1, \dots, z_r]$

  $A := \mathrm{Kos}_Q(z_1^p, \dots, z_r^p)$

<u>Lemma</u>: $H_i(A) = \begin{cases} R & i = 0 \\ 0 & i > 0. \end{cases}$

<u>Proof</u> : Set $A_j = \mathrm{Kos}_Q(z_1^p, \dots, z_j^p)$

Claim: $H_i(A_j) = \begin{cases} Q/(z_1^p, \dots, z_j^p) & i=0 \\ 0 & i>0 \end{cases}$

True for $j=1$: $A_1 = (0 \to Q \xrightarrow{z_1^p} Q \to 0)$

Assume true for some j, then:

$$H_i(A_{j+1}) = \mathrm{Tor}_i^Q\left(Q/(z_1^p, \dots, z_j^p), Q/(z_{j+1}^p)\right) =$$

$$= \begin{cases} Q/(z_1^p, \dots, z_{j+1}^p) & i=0 \\ 0 & i>0. \end{cases}$$

QED.

This holds because: the sequence $(z_1^p, \dots, z_r^p)$ is regular!

- $A \xrightarrow{\simeq} R$ quasi-isom. of dg-algebras
  $\Rightarrow$ get equivalence of derived cats
  $$D^f(A) \rightleftarrows D^f(R) \quad \text{(as explained by Ivo).}$$

- Now, another Koszul complex:
  Let $K := \mathrm{Kos}_Q(z_1, \dots, z_r)$
  Then: $\boxed{K \xrightarrow{\simeq} k}$ (a "homological thickening of k")

<u>Key step</u>: $A \to A \otimes K$

have $\quad D^f(R) \simeq D^f(A) \rightleftarrows D^f(A \otimes K)$

not an equiv.! But:

$$\boxed{\text{Prop: There is an isomorphism of lattices: } \left\{ \begin{matrix} \text{thick subcats} \\ \text{of } D^f(A) \end{matrix} \right\} \overset{\sim}{\rightleftarrows} \left\{ \begin{matrix} \text{thick subcats} \\ \text{of } D^f(A \otimes K) \end{matrix} \right\}}$$

~~Proof~~:

Proof: - $N \in D^f(A \otimes K)$, ~~$N \otimes_K \ldots$~~, counit:

$$\bigoplus_{i=1}^{r} \Lambda^i N[i] \underset{(***)}{\cong} N \otimes_A (A \otimes K) \to N$$

$\rightsquigarrow$ they generate the same thick subcats.

$\rightsquigarrow$ get an injection $\hookleftarrow$ of lattices.

- $M \in D^f(A)$, unit: $M \longrightarrow M \otimes K$

$$\begin{matrix} M & \longrightarrow & M \otimes K \\ \| & & \| \\ R \otimes^L_R M & & \underbrace{(R \otimes^L_Q K)}_{\bar{K}} \otimes^L_K M \end{matrix}$$

Hence, enough to show that:

$$\mathrm{Thick}(R) = \mathrm{Thick}(\bar{K})$$

$\supseteq$: clear

$\subseteq$: because R is zero-dim. local ring!

$\Rightarrow \operatorname{Supp} R = \{m\}$

$\operatorname{Supp} \bar{K} = \{m\}$ too!

$$\xRightarrow{\text{Hopkins-Neeman}} \mathrm{Thick}(R) = \mathrm{Thick}(\bar{K}).$$

QED!

(***): $K \otimes_Q k \overset{K \to k}{\cong} K \otimes_Q k = \otimes(\cancel{k} \xrightarrow{0} \cancel{k}) = \bigoplus \Sigma^i k^{\binom{n}{i}}$

and ${}_{A \otimes K}N = {}_A N_K \cong {}_A N \otimes_K K \rightsquigarrow$ have a double: $K \otimes_k K$

$$A \otimes_Q K \xrightarrow{\simeq} A \otimes_Q k = \Lambda^\bullet(k^r[1])$$

kills the diff'l

$$\Longrightarrow \quad D^f(A \otimes K) \xrightarrow{\simeq} D^f(\Lambda) \cong D^f(S)$$

dg-version of BGG
(Andrea's talk)

$\rightarrow$ steps 2 & 3.

(Actually: the K there is the $\bar{K}$ …
because $A \otimes K \xrightarrow{\simeq} R \otimes K \underset{\text{def.}}{=} \bar{K}$)

Now: the arrows work as wanted, i.e.:
describe the maps V & T.

$$D^f(R) \xrightarrow{\simeq} D^f(A) \longrightarrow D^f(A \otimes K) \xrightarrow{\simeq} D^f(\Lambda) \simeq D^f(S)$$

$$k \longmapsto k \longmapsto k \otimes K$$

$$\mathrm{Ext}_R(k,k) \simeq \mathrm{Ext}_A(k,k) \overset{\text{*r}}{=} k[\xi_1, \dots, \xi_r, \theta_1, \dots, \theta_r]$$

$$\uparrow$$

$$\mathrm{Ext}_{A \otimes K}(k,k)$$
$$\|$$
$$\mathrm{Ext}_\Lambda(k,k) = k[\theta_1, \dots, \theta_r]$$

Recalculate (**): by a semi-free resolution:

$$\begin{tikzcd} K \arrow[r, "\simeq"] & k \\ A \otimes \tilde{S}^* \otimes K \arrow[u, "\simeq"] \arrow[ur, dashed] & \end{tikzcd}$$

, but K not semi-free / A

$\tilde{S} := Q[\theta_1, \dots, \theta_r]$, $|\theta_i| = -2$.

semi-free res. of k / A. [Also: semi-free over $A \otimes K$!]

$$\Rightarrow \mathrm{Hom}_A(A \otimes \tilde{S}^* \otimes K, k)$$

, enough to know where the generators go!

$$\cong \mathrm{Hom}(\tilde{S}^* \otimes K, k)$$

$$\cong (\tilde{S} \otimes k) \otimes \mathrm{Hom}_k(\Lambda, k) = k[\theta_1, \dots, \theta_r, \xi_1, \dots, \xi_r]$$

$$\mathrm{Hom}_{A \otimes K}(A \otimes \tilde{S}^* \otimes K, k) \cong \mathrm{Hom}_Q(\tilde{S}^*, k) = k[\theta_1, \dots, \theta_r]$$

$$= \mathrm{Ext}_{A \otimes k}(k, k) \qquad = S$$

$$D^f(R) \longrightarrow D^f(\Lambda) \overset{BGG}{\cong} D^f(S)$$

$$M \longmapsto (A \otimes k) \otimes M = \tilde{M} \longmapsto R\mathrm{Hom}_\Lambda(k, \tilde{M})$$

homology

thus classifies by Hopkins-Neeman $\rightarrow$ $\mathrm{Supp}_S \, \mathrm{Ext}_\Lambda(k, \tilde{M}) \subseteq \mathrm{Spec}(S)$

$\cong \mathrm{Ext}_R(k, M)$ (up to dir. summands, perhaps...)

That's basically it ...

supp kM